

Anharmonic nonlinear oscillator

a. Forces:
$$\begin{cases} \tilde{F} = -e E_1 \tilde{e}^{i\omega t} \\ \tilde{F}_0 = -k_1 \tilde{x} - k_2 \tilde{x}^2 \quad (\text{we use order 2 only}) \\ \tilde{F}_d = -2\gamma m \dot{\tilde{x}} \end{cases}$$

 $\underbrace{k_2 \tilde{x}^2}_{\text{nonlinear restoring force introducing anharmonicity}}$

equation of motion:
$$\ddot{\tilde{x}} + 2\gamma \dot{\tilde{x}} + \underbrace{\frac{k_1}{m}}_{\equiv \omega_0^2} \tilde{x} + \underbrace{\frac{k_2}{m}}_{\equiv a} \tilde{x}^2 = - \frac{e E_1 \tilde{e}^{i\omega t}}{m} \quad (1)$$

γ : damping rate

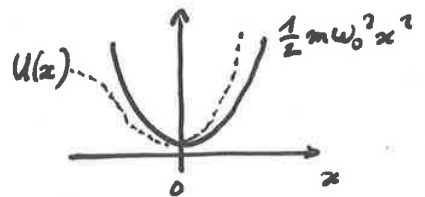
ω_0 : angular frequency of the harmonic oscillator

a : parameter characterizing the strength of the nonlinearity

b.
$$U(\tilde{x}) = - \int \tilde{F}_{\text{restoring}} \cdot d\tilde{x} = - \int (\tilde{F}_0 + \tilde{F}_d) \cdot d\tilde{x}$$

$$= \frac{1}{2} m \omega_0^2 \tilde{x}^2 + \frac{1}{3} m a \tilde{x}^3$$

$\underbrace{\frac{1}{3} m a \tilde{x}^3}_{\text{anharmonicity breaking the energy symmetry}}$



c. [Perturbative approach: driving field $\lambda E_1 \tilde{e}^{i\omega t}$ with $\lambda \in [0, 1]$]
 λ characterizes the strength of the perturbation from driving field.

Equation (1) becomes
$$\ddot{\tilde{x}} + 2\gamma \dot{\tilde{x}} + \omega_0^2 \tilde{x} + a \tilde{x}^2 = - \frac{\lambda e E_1 \tilde{e}^{i\omega t}}{m} \quad (2)$$

We now seek $\tilde{x}$ as a power series expansion in λ :

$$\tilde{x} = \lambda \tilde{x}^{(1)} + \lambda^2 \tilde{x}^{(2)} + \underbrace{(\lambda^3 \tilde{x}^{(3)} + \dots)}_{\text{we neglect this part}}$$

We inject it in (2):

$$\lambda \ddot{\tilde{x}}^{(1)} + \lambda^2 \ddot{\tilde{x}}^{(2)} + 2\gamma [\lambda \dot{\tilde{x}}^{(1)} + \lambda^2 \dot{\tilde{x}}^{(2)}] + \omega_0^2 [\lambda \tilde{x}^{(1)} + \lambda^2 \tilde{x}^{(2)}] + a [\lambda^2 (\tilde{x}^{(1)})^2 + 2\lambda^3 \tilde{x}^{(1)} \tilde{x}^{(2)} + \lambda^4 (\tilde{x}^{(2)})^2] = - \frac{\lambda e E_1 \tilde{e}^{i\omega t}}{m}$$

In order for this equation to be a solution for every λ , it has to satisfy equations with terms proportional to λ and λ^2 separately:

here we say $\lambda = 1$
$$\begin{cases} \textcircled{\lambda} \ddot{\tilde{x}}^{(1)} + 2\gamma \dot{\tilde{x}}^{(1)} + \omega_0^2 \tilde{x}^{(1)} = - \frac{e E_1 \tilde{e}^{i\omega t}}{m} & (3a) \\ \textcircled{\lambda^2} \ddot{\tilde{x}}^{(2)} + 2\gamma \dot{\tilde{x}}^{(2)} + \omega_0^2 \tilde{x}^{(2)} + a (\tilde{x}^{(1)})^2 = 0 & (3b) \end{cases}$$

In order to solve this we need to use the following:

$$\text{ANSATZ } \tilde{x}^{(1)}(t) = x^{(1)}(\omega_1) e^{-i\omega_1 t} + x^{(1)}(\omega_2) e^{-i\omega_2 t} + \dots$$

$$= \sum_n x^{(1)}(\omega_n) e^{-i\omega_n t}$$

Note that in our case, the driving field contains only one frequency, thus reducing the number of frequencies in $\tilde{x}(t)$.

$$(3a) \quad -\omega_n^2 x^{(1)} e^{-i\omega_n t} - 2\gamma i\omega_n x^{(1)} e^{-i\omega_n t} + \omega_0^2 x^{(1)} e^{-i\omega_n t} = -\frac{eE_1 e^{-i\omega t}}{m}$$

Here the driving field produces only a response at frequency ω .

$$\boxed{x^{(1)}(\omega) = \frac{-eE_1}{m(\omega_0^2 - 2\gamma i\omega - \omega^2)} = -\frac{e}{m} \frac{E_1}{D(\omega)}}$$

$$P^{(1)}(\omega) = -Ne x^{(1)}(\omega) = \frac{Ne^2 E_1}{m D(\omega)} = \epsilon_0 \chi^{(1)}(\omega; \omega) E_1(\omega)$$

d. Same reasoning with (3b) equation:

$$-\omega_n^2 x^{(2)} e^{-i\omega_n t} - 2\gamma i\omega_n x^{(2)} e^{-i\omega_n t} + \omega_0^2 x^{(2)} e^{-i\omega_n t} + a x^{(1)}(\omega_l) e^{-i\omega_l t} = x^{(2)}(\omega_m) e^{-i\omega_m t} = 0$$

Here the driving field imposes $\omega_l = \omega_m = \omega$, thus $\omega_n = 2\omega$ to satisfy the equation.

$$\text{then } \boxed{x^{(2)}(2\omega) = \frac{a(-eE_1/m D(\omega))^2}{D(2\omega)} = \frac{ae^2 E_1^2}{m^2 D(2\omega) D(\omega)^2}} \quad \text{generation of second harmonic}$$

$$P^{(2)}(2\omega) = \epsilon_0 \chi^{(2)}(2\omega; \omega, \omega) E_1(\omega)^2 = \frac{-Ne^3 a E_1^2}{m^2 D(2\omega) D(\omega)^2}$$

$$\chi^{(2)}(2\omega; \omega, \omega) = -\frac{Ne^3 a}{\epsilon_0 m^2 D(2\omega) D(\omega)^2}$$

e. Non-resonant condition: $\omega_0 \gg \omega$

$$\text{Then } \boxed{\chi^{(2)}(2\omega; \omega, \omega) \sim -\frac{Na e^3}{\epsilon_0 m^2 \omega_0^6}}$$

f. We are in 1D so we can use $\chi_{333}^{(2)}$ as approximation for $\chi^{(2)}$.

$$a = -\frac{\chi^{(2)} \epsilon_0 m^2 \omega_0^6}{Ne^3} \quad \left| \frac{a}{\omega_0^2} \right| \sim$$